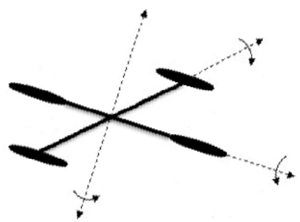
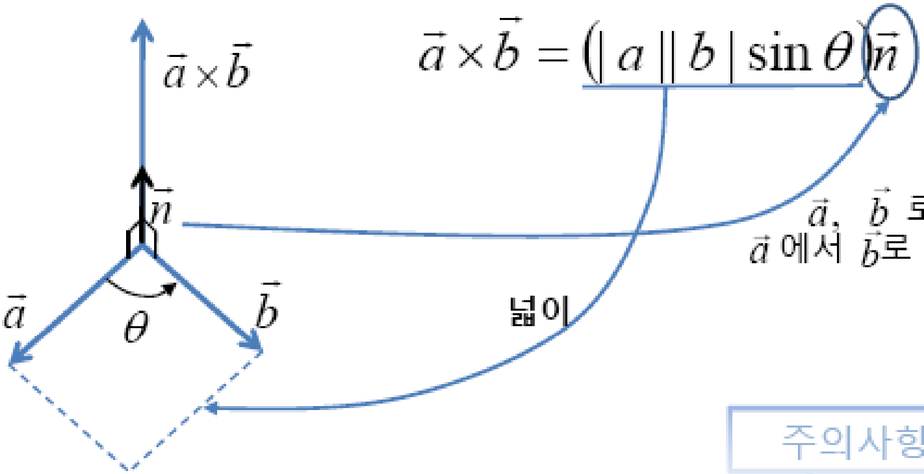
7장 Vectors

7.4 외적



- 내적의 정의

$$\vec{a} \times \vec{b} = (|a||b|\sin\theta)\vec{n}$$



$\vec{a}$, $\vec{b}$ 로 이루어진 평면의 단위법선벡터.
$\vec{a}$ 에서 $\vec{b}$로 오른손을 감싸 쥘 때 엄지의 방향.

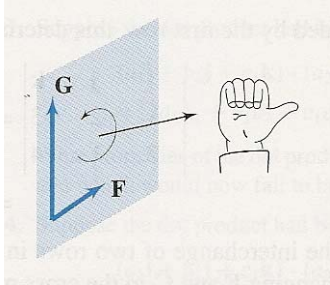
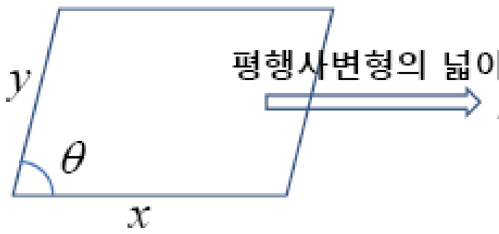


FIGURE 5.24 *Right-hand rule gives the direction of* **F** × **G**.

기초지식



$$S = xy\sin\theta$$

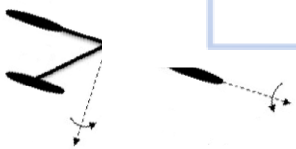
주의사항

벡터의 외적은 교환법칙, 결합 법칙이 성립하지 않는다.

$$\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$$

$$\because \sin(-\theta) = -\sin\theta$$

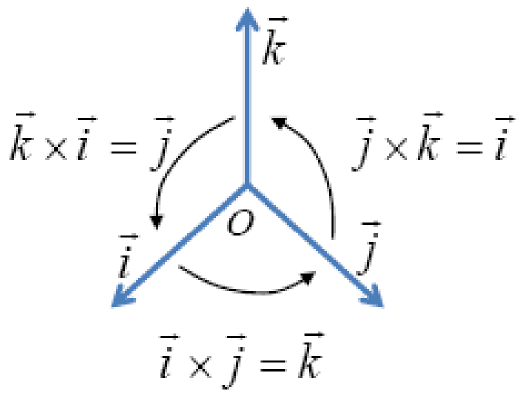
$$\vec{a} \times (\vec{b} \times \vec{c}) \neq (\vec{a} \times \vec{b}) \times \vec{c}$$



- 단위 벡터의 외적과 벡터 외적의 행렬식 표현

$\vec{a}\times\vec{b}=(|a||b|\sin\theta)\vec{n}$

$\vec{a}\times\vec{b}=-\vec{b}\times\vec{a}$



$$\vec{i}\times\vec{j}$$
$$=(|\vec{i}||\vec{j}|\sin 90^\circ)\vec{k}$$
$$=\vec{k}$$

벡터외적의 분배법칙

$$\vec{a}\times(\vec{b}+\vec{c})=\vec{a}\times\vec{b}+\vec{a}\times\vec{c}$$

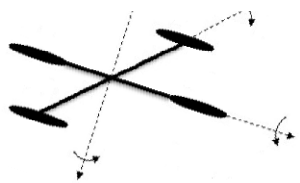
$\vec{a}=[a_1,a_2,a_3],\ \vec{b}=[b_1,b_2,b_3]$

$$\vec{a}=a_1\vec{i}+a_2\vec{j}+a_3\vec{k},\quad \vec{b}=b_1\vec{i}+b_2\vec{j}+b_3\vec{k}$$

$$\vec{a}\times\vec{b}=\left(a_1\vec{i}+a_2\vec{j}+a_3\vec{k}\right)\times\left(b_1\vec{i}+b_2\vec{j}+b_3\vec{k}\right)$$
$$=a_1b_1\vec{i}\times\vec{i}+a_1b_2\vec{i}\times\vec{j}+a_1b_3\vec{i}\times\vec{k}$$
$$+a_2b_1\vec{j}\times\vec{i}+a_2b_2\vec{j}\times\vec{j}+a_2b_3\vec{j}\times\vec{k}$$
$$+a_3b_1\vec{k}\times\vec{i}+a_3b_2\vec{k}\times\vec{j}+a_3b_3\vec{k}\times\vec{k}$$
$$=(a_2b_3-a_3b_2)\vec{i}+(a_3b_1-a_1b_3)\vec{j}+(a_1b_2-a_2b_1)\vec{k}$$
$$=\begin{vmatrix}a_2 & a_3\\ b_2 & b_3\end{vmatrix}\vec{i}+\begin{vmatrix}a_3 & a_1\\ b_3 & b_1\end{vmatrix}\vec{j}+\begin{vmatrix}a_1 & a_2\\ b_1 & b_2\end{vmatrix}\vec{k}$$

$$\vec{a}\times\vec{b}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}$$

⇦ 행렬의 크기



$$\mathbf{a}\times\mathbf{b}=(a_2b_3-a_3b_2)\mathbf{i}-(a_1b_3-a_3b_1)\mathbf{j}+(a_1b_2-a_2b_1)\mathbf{k}$$

(*i*) $\mathbf{a}=\mathbf{0}$ 또는 $\mathbf{b}=\mathbf{0}$ 이면 $\mathbf{a}\times\mathbf{b}=\mathbf{0}$

(*ii*) $\mathbf{a}\times\mathbf{b}=-\mathbf{b}\times\mathbf{a}$

(*iii*) $\mathbf{a}\times(\mathbf{b}+\mathbf{c})=(\mathbf{a}\times\mathbf{b})+(\mathbf{a}\times\mathbf{c})$

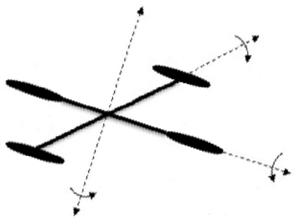
(*iv*) $(\mathbf{a}+\mathbf{b})\times\mathbf{c}=(\mathbf{a}\times\mathbf{c})+(\mathbf{b}\times\mathbf{c})$ (분배법칙)

(*v*) $\mathbf{a}\times(k\mathbf{b})=(k\mathbf{a})\times\mathbf{b}=k(\mathbf{a}\times\mathbf{b})$ (k는 스칼라)

(*vi*) $\mathbf{a}\times\mathbf{a}=\mathbf{0}$

(*vii*) $\mathbf{a}\cdot(\mathbf{a}\times\mathbf{b})=0$

(*viii*) $\mathbf{b}\cdot(\mathbf{a}\times\mathbf{b})=0$



정리 7.2 **평행인 벡터들에 대한 판정**

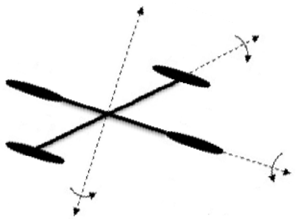
$\mathbf{0}$이 아닌 두 벡터 $\mathbf{a}$와 $\mathbf{b}$가 평행이 되기 위한 필요충분조건은 $\mathbf{a}\times\mathbf{b}=\mathbf{0}$이다.

예제 2 평행인 벡터들

(a) 성질 (*vi*)으로부터 다음을 얻는다.

$$\mathbf{i}\times\mathbf{i}=\mathbf{0},\quad \mathbf{j}\times\mathbf{j}=\mathbf{0},\quad \mathbf{k}\times\mathbf{k}=\mathbf{0} \tag{2}$$

(b) $\mathbf{a}=2\mathbf{i}+\mathbf{j}-\mathbf{k}$, $\mathbf{b}=-6\mathbf{i}-3\mathbf{j}+3\mathbf{k}=-3\mathbf{a}$이면, $\mathbf{a}$와 $\mathbf{b}$는 평행이다. 따라서 정리 7.2로부터 $\mathbf{a}\times\mathbf{b}=\mathbf{0}$이다. 이 결과는 또한 성질 (*v*)와 (*vi*)을 결합시킴으로써 나온다. ❑



예제 4 외적

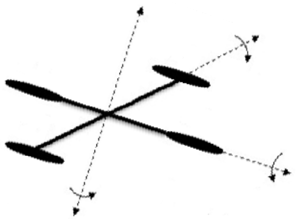
$\mathbf{a}=4\mathbf{i}-2\mathbf{j}+5\mathbf{k}$, $\mathbf{b}=3\mathbf{i}+\mathbf{j}-\mathbf{k}$ 라 할 때, $\mathbf{a}\times\mathbf{b}$ 를 구하라.

풀이 (8)에서 다음 식을 얻는다.

$$\mathbf{a}\times\mathbf{b}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 4 & -2 & 5\\ 3 & 1 & -1\end{vmatrix}=\begin{vmatrix}-2 & 5\\ 1 & -1\end{vmatrix}\mathbf{i}-\begin{vmatrix}4 & 5\\ 3 & -1\end{vmatrix}\mathbf{j}+\begin{vmatrix}4 & -2\\ 3 & 1\end{vmatrix}\mathbf{k}$$

$$= -3\mathbf{i}+19\mathbf{j}+10\mathbf{k}$$

❑



EXAMPLE 5.11

Suppose we want the equation of the plane Π containing the points $(1, 2, 1)$, $(-1, 1, 3)$, and $(-2, -2, -2)$.

Begin by finding a vector normal to Π. We will do this by finding two vectors in Π and taking their cross product. The vectors from $(1, 2, 1)$ to the other two given points are in Π (Figure 5.25). These vectors are

$$\mathbf{F} = -2\mathbf{i} - \mathbf{j} + 2\mathbf{k} \quad \text{and} \quad \mathbf{G} = -3\mathbf{i} - 4\mathbf{j} - 3\mathbf{k}.$$

Form

$$\mathbf{N} = \mathbf{F} \times \mathbf{G} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & -1 & 2 \\ -3 & -4 & -3 \end{vmatrix} = 11\mathbf{i} - 12\mathbf{j} + 5\mathbf{k}.$$

This vector is normal to Π (orthogonal to every vector lying in Π). Now proceed as in Example 5.10. If (x, y, z) is any point in Π, then $(x-1)\mathbf{i} + (y-2)\mathbf{j} + (z-1)\mathbf{k}$ is in Π and so is orthogonal to $\mathbf{N}$. Therefore,

$$[(x-1)\mathbf{i} + (y-2)\mathbf{j} + (z-1)\mathbf{k}] \cdot \mathbf{N} = 11(x-1) - 12(y-2) + 5(z-1) = 0.$$

This gives

$$11x - 12y + 5z = -8.$$

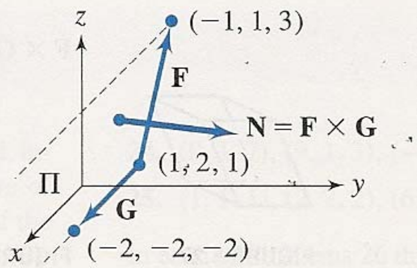
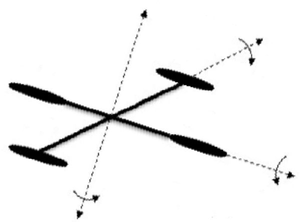


FIGURE 5.25



8. 스칼라 삼중적 (Scalar triple product)

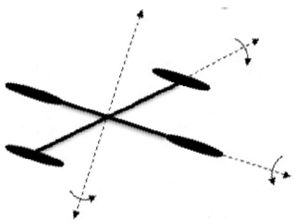
$\vec{a} = [a_1,\ a_2,\ a_3],\ \vec{b} = [b_1,\ b_2,\ b_3],\ \vec{c} = [c_1,\ c_2,\ c_3]$

$$(\vec{a}, \vec{b}, \vec{c}) = \vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

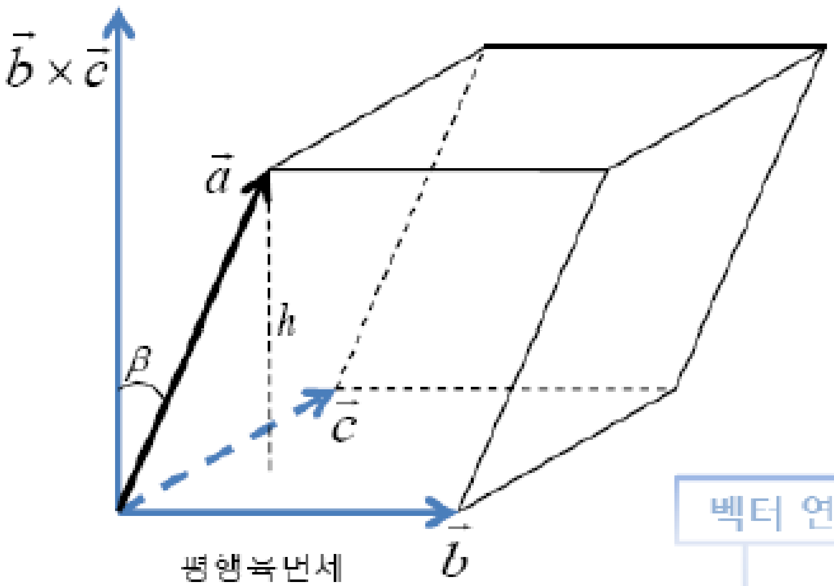
$$\vec{a} \cdot (\vec{b} \times \vec{c}) = (a_1\vec{i} + a_2\vec{j} + a_3\vec{k}) \cdot \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$- (a_1\vec{i} + a_2\vec{j} + a_3\vec{k}) \cdot \left(\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} \vec{i} - \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} \vec{j} + \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} \vec{k} \right)$$

$$= a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$



9. 스칼라 삼중적의 기하학적인 해석



$$\vec{a}\cdot(\vec{b}\times\vec{c}) = \left|\vec{b}\times\vec{c}\right|\left|\vec{a}\right|\cos\beta$$

$\left|\vec{b}\times\vec{c}\right|$ → 밑면의 넓이

$\left|\vec{a}\right|\cos\beta$ → h

$\vec{a}\cdot(\vec{b}\times\vec{c})$: 평행육면체의 부피

벡터 연산의 기하학적 의미

- 내적 → 길이
- 외적 → 넓이
- 삼중적 → 부피

예제 5 삼각형의 면적

점 $P_1(1, 1, 1)$, $P_2(2, 3, 4)$와 $P_3(3, 0, -1)$에 의하여 결정되는 삼각형의 면적을 구하라.

풀이 벡터 $\overrightarrow{P_1P_2}$와 $\overrightarrow{P_2P_3}$는 삼각형의 두 변으로 취급할 수 있다. $\overrightarrow{P_1P_2} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$, $\overrightarrow{P_2P_3} = \mathbf{i} - 3\mathbf{j} - 5\mathbf{k}$ 이므로

$$\overrightarrow{P_1P_2} \times \overrightarrow{P_2P_3} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ 1 & -3 & -5 \end{vmatrix} = \begin{vmatrix} 2 & 3 \\ -3 & -5 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & 3 \\ 1 & -5 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & 2 \\ 1 & -3 \end{vmatrix}\mathbf{k}$$

$$= -\mathbf{i} + 8\mathbf{j} - 5\mathbf{k}$$

이다. (12)로부터 면적은

$$A = \frac{1}{2}\|-\mathbf{i} + 8\mathbf{j} - 5\mathbf{k}\| = \frac{3}{2}\sqrt{10}$$

이다. ❑

EXAMPLE 5.13

One corner of a rectangular parallelepiped is at $(-1, 2, 2)$, and three incident sides extend from this point to $(0, 1, 1)$, $(-4, 6, 8)$, and $(-3, -2, 4)$. To find the volume of this solid, form the vectors

$$\mathbf{F} = (0 - (-1))\mathbf{i} + (1 - 2)\mathbf{j} + (1 - 2)\mathbf{k} = \mathbf{i} - \mathbf{j} - \mathbf{k},$$

$$\mathbf{G} = (-4 - (-1))\mathbf{i} + (6 - 2)\mathbf{j} + (8 - 2)\mathbf{k} = -3\mathbf{i} + 4\mathbf{j} + 6\mathbf{k},$$

and

$$\mathbf{H} = (-3 - (-1))\mathbf{i} + (-2 - 2)\mathbf{j} + (4 - 2)\mathbf{k} = -2\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}.$$

Calculate

$$\mathbf{F} \times \mathbf{G} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & -1 \\ -3 & 4 & 6 \end{vmatrix} = -2\mathbf{i} - 3\mathbf{j} + \mathbf{k}.$$

Then,

$$\mathbf{H} \cdot (\mathbf{F} \times \mathbf{G}) = (-2)(-2) + (-4)(-3) + (2)(1) = 18,$$

and the volume is 18 cubic units. ■